IOPscience iopscience.iop.org

Home Search Collections Journals About Contact us My IOPscience

On the superdeterminant function for supermatrices

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1984 J. Phys. A: Math. Gen. 17 1389

(http://iopscience.iop.org/0305-4470/17/7/006)

View the table of contents for this issue, or go to the journal homepage for more

Download details:
IP Address: 129.252.86.83
The article was downloaded on 31/05/2010 at 08:29

Please note that terms and conditions apply.

J. Phys. A: Math. Gen. **17** (1984) 1389–1395. Printed in Great Britain

On the superdeterminant function for supermatrices

N B Backhouse and A G Fellouris†

Department of Applied Mathematics and Theoretical Physics, The University, PO Box 147, Liverpool L69 3BX, UK

Received 3 November 1983

Abstract. A rigorous proof is given of the multiplicative property of the superdeterminant (Berezinian) of a supermatrix. The proof obviates the need to consider the domain of existence of the logarithm function for supermatrices, and devolves on the identity $\det(1-PQ)\det(1-QP)=1$, where P and Q are compatible rectangular matrices over the odd part of a Grassmann algebra.

1. Introduction

An important construction in any supersymmetric field theory is that of invariant integrals on the underlying curved superspace or supermanifold. Arnowitt *et al* (1975) show that this required the concept of a superspace scalar density and a determinant function for the supermatrices of linear superspace transformations. This determinant function, subsequently called the superdeterminant, was independently discovered by Berezin and Leites (1975). Our principal aim in this paper is to prove rigorously that the superdeterminant is a multiplicative function, and in order to do this we need some basic results from the theory of Grassmann algebras. Let $\boldsymbol{G}_p$ denote the Grassmann algebra on $p<\infty$ mutually anticommuting generators. $\boldsymbol{G}_p$ can be written as the direct sum of two subspaces $\boldsymbol{G}_{p,0}\oplus\boldsymbol{G}_{p,1}$, where $\boldsymbol{G}_{p,0}$ (resp. $\boldsymbol{G}_{p,1}$) is the even (resp. odd) part of $\boldsymbol{G}_p$ and consists of all linear combinations of products of an even (resp. odd) number of generators. $\boldsymbol{G}_{p,0}$ contains the identity 1, regarded as the product of a zero number of generators. The elements of $\boldsymbol{G}_{p,0}$ commute with the elements of $\boldsymbol{G}_p$, whereas the elements of $\boldsymbol{G}_{p,1}$ mutually anticommute. There is an alternative decomposition $\boldsymbol{G}_p=\boldsymbol{F}1\oplus\boldsymbol{N}_p$, where $\boldsymbol{F}$ is the field of scalars (i.e. the real or complex numbers) and $\boldsymbol{N}_p$ consists of all linear combinations of a non-zero number of generators. We call $\boldsymbol{F}1$ the numeric component of $\boldsymbol{G}_p$. The elements of $\boldsymbol{N}_p$ are nilpotent and have degree of nilpotency $\leqslant p$. The invertible elements of $\boldsymbol{G}_p$ are of the form $f1+n$, where $f\neq 0$ and $n\in\boldsymbol{N}_p$, and constitute a subgroup $\boldsymbol{G}_p^*$ of $\boldsymbol{G}_p$.

These are the principal algebraic properties of $\boldsymbol{G}_p$ which we shall use. Proofs and further details can be found under the heading of Grassmann or exterior algebra in many standard text books on algebra—for example Mostow *et al* (1963).

Rogers (1980) has shown that $\boldsymbol{G}_p$ has a norm which gives it the structure of a Banach algebra and enables one to do analysis. Rogers (1980) further shows that p, the cardinality of the set of generators, can also be taken to be that of the set of all integers. In that case $\boldsymbol{G}_\infty$ consists of all those linear combinations of products, a finite

† On leave of absence from the Department of Mathematics, National Technical University, Athens, Greece.

0305-4470/84/071389+07$02.25 © 1984 The Institute of Physics

number of factors in each product, from a countable set of anticommuting generators, which have finite norm. $\boldsymbol{G}_\infty$ is a Banach algebra which retains some, but not all, of the algebraic properties of $\boldsymbol{G}_p$ for $p<\infty$. For example, the elements of $\boldsymbol{G}_{\infty,0}$ commute with all of $\boldsymbol{G}_\infty$, and the elements of $\boldsymbol{G}_{\infty,1}$ mutually anticommute. In particular the elements of $\boldsymbol{G}_{\infty,1}$ are nilpotent of degree two. In general, however, the elements of $\boldsymbol{N}_\infty$ are not nilpotent in the algebraic sense—see Rogers (1980), lemma 2.7, for the sense in which the elements of $\boldsymbol{N}_\infty$ are topologically nilpotent.

A supermatrix over $\boldsymbol{G}_p$, $p\leqslant\infty$, of dimension $(m+n)\times(m+n)$ is a matrix of the form

$$M=\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right), \tag{1}$$

where A (resp. D) is an $m\times m$ (resp. $n\times n$) matrix whose entries belong to $\boldsymbol{G}_{p,0}$, and B (resp. C) is an $m\times n$ (resp. $n\times m$) matrix whose entries belong to $\boldsymbol{G}_{p,1}$. The set of all such matrices form an associative algebra over $\boldsymbol{F}$ under the usual matrix addition and multiplication. It is known from the work of van Nieuwenhuizen (1981) and Ebner (1982) that M is invertible if and only if A and D are invertible, which is the case if and only if the matrices formed from the numeric components of the entries in A and D are invertible. Explicitly

$$M^{-1}=\left(\begin{array}{c|c} A^{-1}+A^{-1}B(D-CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D-CA^{-1}B)^{-1} \\ \hline -(D-CA^{-1}B)^{-1}CA^{-1} & (D-CA^{-1}B)^{-1} \end{array}\right) \tag{2}$$

$$=\left(\begin{array}{c|c} (A-BD^{-1}C)^{-1} & -(A-BD^{-1}C)^{-1}BD^{-1} \\ \hline -D^{-1}C(A-BD^{-1}C)^{-1} & D^{-1}C(A-BD^{-1}C)^{-1}BD^{-1}+D^{-1} \end{array}\right). \tag{3}$$

These two forms of M^{-1} look different at first sight, but can easily be shown to be equivalent.

There is a linear $\boldsymbol{G}_{p,0}$-valued function on the set of all supermatrices, called the supertrace, and defined by

$$\operatorname{str} M=\operatorname{tr} A-\operatorname{tr} D, \tag{4}$$

where the trace function is the usual sum of diagonal elements for a square matrix. This was first defined by Arnowitt *et al* (1975).

There is another standard $\boldsymbol{G}_{p,0}$-valued function, called the superdeterminant or Berezinian, and defined by

$$\operatorname{sdet} M=(\det A)\det{}^{-1}(D-CA^{-1}B), \tag{5}$$

for invertible supermatrices. $\boldsymbol{G}_{p,0}$ is a commutative algebra, so the determinants appearing in (5) are well defined.

It is the main purpose of this paper to provide a rigorous proof that the superdeterminant is a multiplicative function. This property has been noted many times and subsequently assumed to be true, for example by Arnowitt *et al* (1975) and van Nieuwenhuizen (1981), who provide a proof which is only valid for a restricted class of supermatrices. It is worth saying that the superdeterminant is essential for the definition and study of a number of supergroups of great importance in supersymmetry—for example see Rittenberg (1978). We feel, therefore, that there is a need to tidy up this area of supermatrix theory.

The argument of Arnowitt *et al* (1975) and van Nieuwenhuizen (1981) that the superdeterminant is multiplicative is based on their alternative definition

$$\operatorname{sdet} M = \exp(\operatorname{str} \ln M), \tag{6}$$

where $\ln M$ is the natural logarithm of M. Our unease with a proof based on (6) is that, in contrast to the exponential, the logarithm is not a well defined function. This is so even for ordinary matrices, except in a neighbourhood of the identity where one can use a power series to define the logarithm and then prove that it is the inverse of the exponential. In this connection we may refer to Curtis (1979) for some comments on the logarithm of a matrix.

Our proof is based directly on the definition (5) and makes use of the remarkable identity $\det(1-PQ)\det(1-QP)=1$, where P and Q are compatible rectangular matrices over $\boldsymbol{G}_{p,1}$. This identity is proved in § 2 using the logarithm function, but in this case we are able to establish its existence rigorously. The application to the superdeterminant function is made in § 3.

2. The identity $\det(1-PQ)\det(1-QP)=1$

We establish a number of preliminary lemmas.

Lemma 1. Let K be an $r\times r$ matrix over $\boldsymbol{G}_{p,0}$ where entries are finite linear combinations of products of pairs of elements of $\boldsymbol{G}_{p,1}$. Then $\ln(1-K)$ is a well defined matrix over $\boldsymbol{G}_{p,0}$ and satisfies

$$\exp\{\ln(1-K)\} = 1-K. \tag{7}$$

Proof. The restriction on the matrix entries implies that K is nilpotent even if $p=\infty$. Then the matrix

$$L = \ln(1-K) = -(K+\tfrac{1}{2}K^2+\ldots K^{k-1}/(k-1)), \tag{8}$$

where $K^k=0$, is well defined. Evidently, L itself is nilpotent of degree $\leqslant k$. Now

$$\exp\{\ln(1-x)\} = 1-x \tag{9}$$

is an identity in formal power series in the indeterminate x, where exp and ln are given by the usual power series. It follows that

$$\exp L = 1-K \tag{10}$$

is valid, where questions of convergence are avoided by noting that both K and L are nilpotent.

Lemma 2. Let L be a matrix over $\boldsymbol{G}_{p,0}$. Then

$$\det\{\exp L\} = \exp\{\operatorname{tr} L\}. \tag{11}$$

Proof. In the special case that L is an $r\times r$ real or complex matrix, the equality (11) is a well known result from matrix algebra and can be proved by reduction to triangular form. For ordinary matrices (11) has two interpretations. On the one hand, if L is a particular matrix, $\det\{\exp L\}$ and $\exp\{\operatorname{tr} L\}$ are scalars, which happen to be equal, obtained by summing absolutely convergent series. On the other hand, each side of

(11) can be regarded as a formal power series in indeterminates L_{ij}, $1 \leqslant i, j \leqslant r$, the matrix elements of L, and then (11) says that each side is a rearrangement of the other.

Now let us suppose the L_{ij} are particular elements of $\boldsymbol{G}_{p,0}$ and therefore subject to the usual additive and multiplicative rules of arithmetic. Both sides of (11) are absolutely convergent series, in this case within the Banach algebra $\boldsymbol{G}_{p,0}$, which must coincide in value, because, as observed above, each is a rearrangement of the other.

Lemma 3. Let P, Q be $m \times n$ and $n \times m$ matrices, respectively, over $\boldsymbol{G}_{p,1}$. Then

$$\operatorname{tr}(PQ)^s = -\operatorname{tr}(QP)^s, \tag{12}$$

for all integers $s \geqslant 1$.

Proof. The result for $s=1$ is an easy consequence of the fact that elements of $\boldsymbol{G}_{p,1}$ mutually anticommute. For $s>1$ we have

$$\operatorname{tr}(PQ)^s = \operatorname{tr} P\{(QP)^{s-1}Q\} = -\operatorname{tr}\{(QP)^{s-1}Q\}P = -\operatorname{tr}(QP)^s.$$

Theorem 1. Let P, Q be $m \times n$ and $n \times m$ matrices, respectively, over $\boldsymbol{G}_{p,1}$. Then

$$\det(I-PQ)\det(I-QP) = 1. \tag{13}$$

Proof. We note that both PQ and QP satisfy the restrictions placed on the matrix K in lemma 1. Then, combining lemmas 1 and 2, we have

$$\det(I-PQ) = \exp\{\operatorname{tr}\ln(I-PQ)\} = \exp(-x),$$

where

$$x = \operatorname{tr}(PQ) + \tfrac{1}{2}\operatorname{tr}(PQ)^2 + \cdots + (1/(k-1))\operatorname{tr}(PQ)^{k-1}, \tag{14}$$

and

$$\det(1-QP) = \exp(-y),$$

where

$$y = \operatorname{tr}(QP) + \tfrac{1}{2}\operatorname{tr}(QP)^2 + \cdots + [1/(k-1)]\operatorname{tr}(QP)^{k-1}. \tag{15}$$

In (14) and (15) k is the least integer for which both $(PQ)^k$ and $(QP)^k$ are zero. It follows from lemma 3 that $y=-x$. Thus

$$\det(I-PQ)\det(I-QP) = \exp(-x)\exp(x) = \exp(-x+x) = \exp(0) = 1. \tag{16}$$

The deduction of (16) from the previous line is a particular case of a property of the absolutely convergent exponential series for commuting quantities, which is *a fortiori* valid for nilpotent arguments—for example see Curtis (1979).

Corollary. If P is a square matrix over $\boldsymbol{G}_{p,1}$,

$$\det(I-P^2) = 1. \tag{17}$$

Proof. Putting $Q=P$ in theorem 1 we have $[\det(1-P^2)]^2 = 1$. By expanding $\det(I-P^2)$ we see that its numeric part is 1. Thus $\det(I-P^2) \neq -1$.

3. The superdeterminant

To prove the multiplicative property of the superdeterminant, as defined by (5), we first establish an alternative form.

Lemma 4.

$$\operatorname{sdet} M = \det(A - BD^{-1}C)\det(D^{-1}), \tag{18}$$

if M is invertible.

Proof. By (5)

$$\begin{aligned}\operatorname{sdet} M &= (\det A)\det(D - CA^{-1}B)^{-1}\\ &= (\det A)\det(1 - CA^{-1}BD^{-1})^{-1}\det(D^{-1}),\end{aligned}$$

using the multiplicative property of the ordinary determinant,

$$= (\det A)\det(1 - BD^{-1}CA^{-1})\det(D^{-1}),$$

using theorem 1 applied to $P = BD^{-1}$ and $Q = CA^{-1}$,

$$= \det(A - BD^{-1}C)\det(D^{-1}),$$

noting that the values of the determinants lie in $\boldsymbol{G}_{p,0}$ and are therefore commuting numbers.

The expression (18) is the form of the superdeterminant first used by Berezin and Leites (1975). It is worth noting that (18) is equivalent to $\operatorname{sdet} M = (\operatorname{sdet}(M^{-1}))^{-1}$, where we use (3) as the form of M^{-1}. We can now prove a special case of the main result.

Lemma 5. Let M, M', M'' be invertible supermatrices of the form

$$M = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right), \qquad M' = \left(\begin{array}{c|c} A' & 0 \\ \hline C' & D' \end{array}\right), \qquad M'' = \left(\begin{array}{c|c} A'' & B'' \\ \hline 0 & D'' \end{array}\right);$$

then

(i) $\operatorname{sdet} M'M = (\operatorname{sdet} M')(\operatorname{sdet} M)$;

(ii) $\operatorname{sdet} M''M = (\operatorname{sdet} M'')(\operatorname{sdet} M)$.

Proof.

(i) By (5) we have

$$\operatorname{sdet} M' = (\det A')(\det D')^{-1}$$

and

$$\operatorname{sdet} M = (\det A)\det(D - CA^{-1}B)^{-1}.$$

Also

$$M'M = \left(\begin{array}{c|c} A'A & A'B \\ \hline C'A + D'C & C'B + D'D \end{array}\right),$$

so that

$$\begin{aligned}\operatorname{sdet} M'M &= (\det A'A)\det\{C'B+D'D-(C'A+D'C)(A'A)^{-1}(A'B)\}^{-1},\\ &= (\det A')(\det A)\det(D'D-D'CA^{-1}B)^{-1},\\ &= (\operatorname{sdet} M')(\operatorname{sdet} M),\end{aligned}$$

as required.

(ii) The proof is similar, except that we use the form (18) for the superdeterminant.

Theorem 2. The superdeterminant is a multiplicative function on the group of invertible supermatrices.

Proof. Let

$$M=\left(\begin{array}{c|c} A & B \\ \hline C & D\end{array}\right),\qquad M'=\left(\begin{array}{c|c} A' & B' \\ \hline C' & D'\end{array}\right)$$

be invertible supermatrices. We can write $M=XY$, where

$$X=\left(\begin{array}{c|c} A & 0 \\ \hline C & I\end{array}\right),\qquad Y=\left(\begin{array}{c|c} 1 & A^{-1}B \\ \hline 0 & D-CA^{-1}B\end{array}\right).$$

Then

$$\begin{aligned}\operatorname{sdet}(MM') &= \operatorname{sdet}(X(YM'))\\ &= (\operatorname{sdet} X)\operatorname{sdet} YM',\end{aligned}$$

using lemma 5(i),

$$= (\operatorname{sdet} X)(\operatorname{sdet} Y)(\operatorname{sdet} M'),$$

using lemma 5(ii),

$$= (\operatorname{sdet} M)(\operatorname{sdet} M'),$$

as required.

To conclude we prove a result for a special class of supermatrices noted by Rittenberg (1978).

Theorem 3. Let M be an invertible supermatrix having the special form

$$M=\left(\begin{array}{c|c} A & B \\ \hline B & A\end{array}\right).$$

Then $\operatorname{sdet} M=1$.

Proof. By definition

$$\operatorname{sdet} M=(\det A)\det(A-BA^{-1}B)^{-1}=\det\{1-(A^{-1}B)^2\}^{-1}=1,$$

using the corollary to theorem 1.

Acknowledgments

One of us (AGF) wishes to thank the National Technical University of Athens for the leave of absence and the Greek Ministry of National Economy for the financial support during the period of preparation of this paper.

References

Arnowitt R, Nath P and Zumino B 1975 *Phys. Lett.* **56** 81–4
Berezin F and Leites D 1975 *Sov. Math. Doklady* **16** 1218–22
Curtis M 1979 *Matrix Group* (Berlin: Springer)
Ebner D 1982 *Gen. Rel. Grav.* **14** 1001–16
Mostow G D, Sampson J H and Meyer J–P 1963 *Fundamental Structures of Algebra* (New York: McGraw-Hill)
Rittenberg V 1978 *Lecture Notes in Physics* **79** (Berlin: Springer) pp 3–21
Rogers A 1980 *J. Math. Phys.* **21** 1352–65
van Nieuwenhuizen P 1981 *Phys. Rep.* **68**